FLOW OF AN IDEAL GAS IN A SPHERICALLY SYMMETRICAL GRAVITATIONAL FIELD WITH ALLOWANCE FOR RADIANT HEAT CONDUCTION AND RADIANT PRESSURE

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1. Basic equations and dimensionless parameters. The system of gas dynamics equations describing the steady-state motion of an ideal gas with radiation in a spherically symmetrical gravitational field can be written as [1]

$$u\frac{du}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM}{r^2} \tag{1.1}$$

$$\rho u\frac{dE}{dr} = \frac{P}{\rho}u\frac{d\rho}{dr} + \frac{1}{r^2}\frac{d}{dr}\left(\lambda r^2\frac{dT}{dr}\right), \qquad \lambda = \frac{4\sigma cT^3}{3\varkappa\rho} \tag{1.2}$$

$$\rho ur^2 = \mu \tag{1.3}$$

$$E = \frac{3}{2}RT + \frac{\sigma T^4}{\rho}, \qquad P = \rho RT + \frac{\sigma T^4}{3}, \qquad R = \frac{k}{m_p}(2X + 0.75Y + 0.5Z) \tag{1.4}$$

Here P, E, T, ρ are the pressure, specific energy, temperature, and density of the material; r is the radius; u is the velocity; R is the specific gas constant; σ is the radiation energy density constant; λ is the coefficient of thermal conductivity; μ is the mass flux divided by 4π, which is considered constant; c is the speed of light; $\varkappa$ is the opacity; the weight concentrations of hydrogen, helium, and heavy elements are X, Y, Z, respectively (the material is assumed to be completely ionized); k is the Boltzmann constant; m_p is the proton mass. If the opacity is due solely to scattering by the electrons, then $\varkappa = 0.19(1+X)$. In future, we shall carry out our calculations for $\varkappa$ = const. If there are no energy sources in the flux, then system (1.1) to (1.4) can be integrated once. We obtain the system of equations

$$-\mu\left(E + \frac{P}{\rho} - \frac{GM}{r} + \frac{u^2}{2}\right) + \lambda r^2\frac{dT}{dr} = -\frac{L}{4\pi}$$

$$u\frac{du}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM}{r^2}, \qquad \rho ur^2 = \mu \tag{1.5}$$

Here L is the total energy flux. If the flux is negligibly small, then L is the energy transfer due to heat conduction. System (1.5) is derived in [2 and 3]. We shall attempt to find a solution of system (1.5), which satisfies the conditions $\rho = T = 0$ at infinity. Such a solution which also satisfies the conditions at the surface boundary of a star, describes the actual efflux from a star. Making use of the expressions for E and P from (1.4), we rewrite (1.5) as

$$\left(\frac{RT}{\rho} - \frac{\mu^2}{\rho^3 r^4}\right)\frac{d\rho}{dr} = \tag{1.6}$$

$$= \frac{2\mu^2}{\rho^2 r^5} - \left(\frac{4\sigma T^3}{3\rho} + R\right)\left\{\frac{\varkappa\mu T}{cr^2} + \frac{3\varkappa\rho}{4\sigma cr^2T^3}\left[\mu\left(\frac{5}{2}RT - \frac{GM}{r} + \frac{\mu^2}{2\rho^2 r^4}\right) - \frac{L}{4\pi}\right]\right\} - \frac{GM}{r^2}$$

$$\frac{dT}{dr} = \frac{\varkappa\mu T}{cr^2} + \frac{3\varkappa\rho}{4\sigma cr^2T^3}\left[\mu\left(\frac{5}{2}RT - \frac{GM}{r} + \frac{\mu^2}{2\rho^2r^4}\right) - \frac{L}{4\pi}\right] \tag{1.7}$$

From gas dynamics we know [2] that solution (1.6), (1.7) satisfies the condition $\rho = 0$ at infinity provided it passes through the singular point of Eq. (1.6) determined by the relation $RT_k/\rho_k = \mu^2/\rho_k^3 r_k^4$ and by the condition that the right-hand side of Eq. (1.6) equals zero. At this point the velocity is equal to the isothermal speed of sound. The point $T = 0$, $\rho = 0$, $r = \infty$ is likewise a singular point for system (1.6), (1.7). In the limiting case $\varkappa \to \infty$, system (1.6), (1.7) describes adiabatic flow and degenerates into algebraic relations. The two singular points of system (1.6), (1.7) for $\varkappa = \infty$ abruptly become a single point at which the velocity is equal to the adiabatic speed of sound. The solution of system (1.6), (1.7) which satisfies the conditions $\rho = T = 0$ for $r = \infty$ gradually becomes the limiting solution for $\varkappa = \infty$.

Let us convert to the independent variable $x = 1/r$ and introduce the dimensionless variables

$$x^* = x/x_k, \qquad \rho^* = \rho/\rho_k, \qquad T^* = T/T_k \tag{1.8}$$

Substituting variables and making use of the first relation at the critical point, we obtain a system of equations in ρ^*, T^* with the independent variable x^*. This system includes the following dimensionless parameters:

$$A_1 = \frac{4\sigma T_k^3}{3\rho_k R}, \qquad A_2 = \frac{3\varkappa\mu}{4\sigma c}\frac{\rho_k}{r_k}\frac{R}{T_k^3}, \qquad A_3 = \frac{GM}{r_k RT_k}, \qquad A_4 = \frac{3\varkappa L}{16\pi\sigma c}\frac{\rho_k}{r_k T_k^4} \tag{1.9}$$

Since $\mu = \sqrt{RT_k}\,\rho_k r_k^2$, it follows that $A_2 = 3\varkappa R^{3/2}\rho_k^2 r_k/4\sigma c T_k^{5/2}$. The second condition at the critical point $x^* = \rho^* = T^* = 1$ imposes a single constraint on the parameters,

$$A_4 = A_2(3 + A_1 - A_3) + \frac{A_3 - 2}{1 + A_1} \tag{1.10}$$

Thus, the system of equations in dimensionless form can be written as

$$\frac{d\rho}{dx} = \left(\frac{2x^3}{\rho^2} - \left(\frac{A_1T^3}{\rho} + 1\right)\left\{\frac{A_2\rho}{T^3}\left(A_1\frac{T^4}{\rho} + 2.5T - A_3x + 0.5\frac{x^4}{\rho^2}\right) - \right.\right.$$
$$\left.\left. - \frac{\rho}{T^3}\left[A_2(3 + A_1 - A_3) + \frac{A_3 - 2}{1 + A_1}\right]\right\} - A_3\right)\left(\frac{x^4}{\rho^3} - \frac{T}{\rho}\right)^{-1} \tag{1.11}$$

$$\frac{dT}{dx} = -\frac{A_2\rho}{T^3}\left(A_1\frac{T^4}{\rho} + 2.5T - A_3x + 0.5\frac{x^4}{\rho^2}\right) + \frac{\rho}{T^3}\left[A_2(3 + A_1 - A_3) + \frac{A_3 - 2}{1 + A_1}\right] \tag{1.12}$$

where the asterisk has been omitted for simplicity.

Henceforth, unless otherwise stipulated, we shall make use of dimensionless variables only. The solution will satisfy the conditions $x = \rho = T = 0$ given a certain relationship among the parameters A_1, A_2, A_3. Each pair of parameters A_1, A_2 is associated with one A_3 and one solution satisfying the zero conditions at infinity. The solution passing through the singular points $x = T = \rho = 1$ has at this point the expansion

$$T = 1 + \beta_1(1 - x) + \beta_2(1 - x)^2 + \beta_3(1 - x)^3, \qquad \rho = 1 + \alpha_1(1 - x) + \alpha_2(1 - x)^2$$

$$\beta_1 = -\frac{A_3 - 2}{1 + A_1}, \qquad \alpha_1 = \frac{1}{4}\left\{A_2(1 + A_1)^2 + 2\frac{A_3 - 2}{1 + A_1} - 8 - (A_2^2(1 + A_1)^4 + \right.$$
$$\left. + 4\left[4 + 7\left(\frac{A_3 - 2}{1 + A_1}\right)^2 + A_2(A_3 - 2)(4 + 7A_1) - 8\frac{A_3 - 2}{A_1 + 1} - 4A_2(1 + A_1)^2\right]\right)^{1/2}\right\}$$

$$\beta_2 = -\frac{1}{2}\frac{A_3 - 2}{1 + A_1}\left[A_2\left(3A_1 + \frac{3}{2}\right) + 3\frac{A_3 - 2}{1 + A_1}\right] - \frac{1}{2}\left[A_2(1 + A_1) + \frac{A_3 - 2}{1 + A_1}\right]\alpha_1$$

$$\alpha_2 = \left\{3A_2\left(1 + A_1 + \frac{(A_3 - 2)^2}{1 + A_1}\right) - 6 - \left[3A_2\left(2A_1 - 2A_1^2 + \frac{5}{2}\right) + 6\frac{A_3 - 2}{1 + A_1}\right] \times \right.$$
$$\times \left(\frac{A_3 - 2}{1 + A_1}\right)^2 + \left[A_2(A_3 + 2 + 4A_1) - 18 - A_2\left(\frac{11}{2} + 3A_1 - 4A_1^2\right)\frac{A_3 - 2}{1 + A_1} - \right.$$
$$\left. - 3\left(\frac{A_3 - 2}{1 + A_1}\right)^2\right]\alpha_1 + \left[A_2\left(\frac{1}{2} + \frac{3}{2}A_1 + A_1^2\right) - 18 + \frac{A_3 - 2}{1 + A_1}\right]\alpha_1^2 +$$

$$+\left[A_2(1+A_1)\left(\frac{5}{2}+4A_1\right)+3\frac{A_3-2}{1+A_1}\right]\beta_2+\alpha_1(\beta_2-5\alpha_1^2)\Big\}\times$$
$$\times\left[A_2(1+A_1)^2+3\frac{A_3-2}{1+A_1}-12-6\alpha_1\right]^{-1}$$

$$\beta_3=A_2\left(1+\frac{(A_3-2)^2}{1+A_1}\right)-\left[A_2\left(2A_1+\frac{5}{2}\right)+2\frac{A_3-2}{1+A_1}\right]\left(\frac{A_3-2}{1+A_1}\right)^2+$$
$$+\frac{1}{3}\left[A_2(A_3+2)-3\left(\frac{A_3-2}{1+A_1}\right)^2-A_2\left(\frac{11}{2}+3A_1\right)\frac{A_3-2}{1+A_1}\right]\alpha_1+\frac{1}{6}A_2\alpha_1^2+$$
$$+\frac{1}{3}\left[A_2\left(\frac{5}{2}+4A_1\right)+3\frac{A_3-2}{1+A_1}\right]\beta_2-\frac{1}{3}\left[A_2(1+A_1)+\frac{A_3-2}{1+A_1}\right]\alpha_2 \qquad (1.13)$$

The solution passing through the point $x = T = \rho = 0$ admits of several expansions (γ_0 is arbitrary),

$$\rho=\gamma_0x^2,\quad T=\delta_0x^{3/4},\quad \delta_0=\left(\frac{4\gamma_0}{3}\right)^{1/4}\left[\frac{A_3-2}{1+A_1}+A_2(A_1+3-A_3)-\frac{A_2}{2\gamma_0^2}\right]^{1/4} \qquad (1.14)$$

or, if the term in square brackets in (1.14) is equal to zero,

$$\rho=\gamma_0x^2,\ T=\delta_0x,\ \gamma_0=\left(\frac{A_2}{2}\right)^{1/2}\left[\frac{A_3-2}{1+A_1}+A_2(A_1+3-A_3)\right]^{-1/2},\quad \delta_0=(0.5A_2\gamma_0)^{1/4} \qquad (1.15)$$

The question of which of expansions (1.14), (1.15) corresponds to the solution satisfying the zero conditions at infinity and passing through the point $x = \rho = T = 1$ requires further study. In [4] only an expansion of the (1.14) type is considered for $x \to 0$. Conditions (1.13) imply the restriction $A_3 > 2$. This restriction is valid for all A_1, A_2.

2. **Limiting cases.** Let us consider the following limiting cases.

a) Let $A_2 = 0$. This case corresponds to the condition $\varkappa = 0$, i.e. to formally infinite heat conductivity. The system of equations becomes

$$\frac{d\rho}{dx}=\frac{2(x^3/\rho^2-1)+[(A_3-2)/(1+A_1)](\rho/T^3-1)}{x^4/\rho^3-T/\rho},\qquad \frac{dT}{dx}=\frac{A_3-2}{1+A_1}\frac{\rho}{T^3} \qquad (2.1)$$

This system depends on the single parameter $B = (A_3 - 2)/(1 + A_1)$. Proceeding from the singular point $x = T = \rho = 1$ in expansion (1.13) for $A_2 = 0$ and solving system (2.1) numerically, we find that the solution passes through the point $x = T = \rho = 0$ for $B = 0.8186$. Hence we have

$$A_3 = 2 + (1 + A_1)\,0.8186 \qquad (2.2)$$

Formally, this solution has no physical meaning, since it corresponds to the case $L = \infty$ or to infinite heat conductivity. Nevertheless, Formula (2.2) can be used for the approximate determination of A_3 for a large heat conductivity, large thermal fluxes, and small A_2. For large x the required solution has the asymptotic form

$$\rho = 0.303x^3,\quad T = 0.705\ x \qquad (2.3)$$

b) Let $A_2 = \infty$. This case corresponds either to zero heat conductivity or to adiabatic flow. The solution can be written in finite form,

$$A_1\frac{T^4}{\rho}+\frac{5}{2}T-A_3x+\frac{1}{2}\frac{x^4}{\rho^2}=3+A_1-A_3,\qquad \ln\frac{T^{3/2}}{\rho}+A_1\left(\frac{T^3}{\rho}-1\right)=0 \qquad (2.4)$$

The first relation of (2.4) is the Bernoulli equation written in dimensionless form; the second means that entropy is constant over the flow. The solution satisfying the conditions $x = T = \rho = 0$ must pass through the point where the velocity is equal to the adiabatic speed of sound. The conditions of passage through this point are given in [5]. In dimensionless form they are

$$\frac{x_a^2}{\rho_a^2}=\frac{A_3}{2},\quad 1+\frac{(1+A_1T_a^3/\rho_a)^2}{1.5+3A_1T_a^3/\rho_a}=\frac{A_3x_a}{2T_a} \qquad (2.5)$$

Here x_a, T_a, ρ_a are the values of the dimensionless parameters at the point where the velocity is equal to the adiabatic speed of sound, $u_a^2 = \gamma P/\rho$. In deriving (2.5) we made use of the expression for the $\gamma = (\partial \ln P/\partial \ln \rho)_S$ of an ideal gas with radiation (S is the specific entropy),

$$\gamma = \frac{1}{1 + A_1T^3/4\rho}\left[1 + \frac{(1 + A_1T^3/\rho)^2}{1.5 + 3A_1T^3/\rho}\right] \tag{2.6}$$

Together with (2.4) written out for the point x_a, relations (2.5) determine A_3, x_a, T_a, ρ_a as functions of A_1. Sample values of these parameters appear in Table 1.

Table 1

A_1	10^4	$5\cdot10^3$	10^3	700	500	200
A_3	$9.803\cdot10^3$	$4.876\cdot10^3$	$9.588\cdot10^2$	668,0	474.9	187,5
x_a	$2.721\cdot10^{-2}$	$3.469\cdot10^{-2}$	$6.173\cdot10^{-2}$	0.07033	0.07962	0.1122
ρ_a	$6.413\cdot10^{-5}$	$1.309\cdot10^{-4}$	$7.005\cdot10^{-4}$	$1.021\cdot10^{-3}$	$1.458\cdot10^{-3}$	$3.878\cdot10^{-3}$
T_a	$4.002\cdot10^{-2}$	$5.076\cdot10^{-2}$	$8.870\cdot10^{-2}$	0.1005	0.1131	0.1564

A_1	100	50	20	15	10	9.5	9
A_3	93.20	46.84	19.75	15.36	11.055	10.63	10.21
x_a	0.1459	0.1898	0.2663	0.2946	0.3371	0.3425	0.3482
ρ_a	$8.161\cdot10^{-3}$	0.01709	0.04373	0.05768	0.08319	0.08692	0.09098
T_a	0.1997	0.2541	0.3431	0.3742	0.4186	0.4242	0.4301

A_1	8.5	8	7.5	7	6.5	6	5.5
A_3	9.785	9.365	8.946	8.529	8.114	7.701	7.291
x_a	0.3544	0.3611	0.3680	0.3754	0.3833	0.3919	0.4012
ρ_a	0.0954	0.1002	0.1055	0.1114	0.1179	0.1251	0.1331
T_a	0.4362	0.4426	0.4494	0.4566	0.4641	0.4721	0.4804

A_1	5	4.5	4	3.5	3	2	1
A_3	6.883	6.478	6.076	5.677	5.283	4.509	3.767
x_a	0.4112	0.4217	0.4335	0.4460	0.4596	0.4886	0.5074
ρ_a	0.1421	0.1522	0.1637	0.1768	0.1917	0.2275	0.2635
T_a	0.4892	0.4985	0.5082	0.5181	0.5282	0.5456	0.5388

A_1	0.5	0.4	0.3	0.2	0.14	0.13	0.128
A_3	3.412	3.342	3.269	3.192	3.1396	3.130	3.128
x_a	0.4705	0.4399	0.3902	0.3043	0.1023	0.02084	0
ρ_a	0.2471	0.2267	0.1907	0.1329	0.02614	$2.671\cdot10^{-3}$	0
T_a	0.4754	0.4381	0.3817	0.2756	0.09330	0.01957	0

In writing out (2.5) we assumed that in a stream of material satisfying the conditions $x = T = \rho = 0$, the flow at infinity must be supersonic. In this case there exists a point x_a where the adiabatic velocity of sound is attained. However, there exists a class of polytropic flows in the spherically symmetrical gravitational field of a constant mass where passage through the speed of sound is impossible. These are polytropic flows with indices $n > 1.5$. Flow with $n = 1.5$ is degenerate and proceeds at a constant Mach number [6]. Polytropic flow with $n = 1.5$ is limiting as $A_1 \to 0$ for solution (2.4). For $A_1 = 0$ there exists an exact solution satisfying the conditions $A_3 = 3$, $\rho = x^{3/2}$, $T = x$ at infinity. The Mach number in such a flux is 3/5; the velocity is always equal to the isothermal speed of sound. For a sufficiently small $A_1 \neq 0$ there is no passage through the adiabatic speed of sound. This is reflected in the fact that for sufficiently small A_1 there is no solution of algebraic system (2.4), (2.5) which determines x_a, T_a, ρ_a, A_3. Let us find the limiting value of A_{1n} for which the flow becomes supersonic. Formally, as $A_1 \to A_{1n}$ the solution of system (2.4), (2.5) must yield $x_a \to 0$. Then, by virtue of the restriction $A_3 > 2$, we find from the first relation of (2.5) that $\rho_a \to \alpha x_a^{3/2}$, while the second relation of (2.5) is fulfilled for $x_a \to 0$ if $T \to \beta x_a$. Substituting these expansions into (2.4), (2.5) and retaining the leading terms, we obtain

$$A_3 = 3 + A_1, \quad \alpha = \left(\frac{2}{3 + A_1}\right)^{1/2}, \quad \beta = \frac{3}{10}(3 + A_1), \quad 3 + A_1 = 2\left(\frac{5}{3}\right)^{3/4} e^{A_1/2} \tag{2.7}$$

The last relation of (2.7) yields the value $A_{1n} = 0.128$.

For $A_1 < A_{1n}$ we have $A_3 = 3 + A_1$; for $x \to 0$ the asymptotic solution is of the form

$$\rho \approx \alpha x^{3/2}, \quad T \approx \beta x, \quad \beta = \alpha^{2/3} e^{2A_1/3}, \quad \left(\frac{2}{5}\right)^{3/2}\left(3 + A_1 - \frac{1}{2\alpha^2}\right)^{3/2} = \alpha e^{A_1} \tag{2.8}$$

For $A_1 > A_{1n}$ the quantity A_3 is determined by the simultaneous solution of system (2.4), (2.5); asymptotically, the solution for $x \to 0$ is of the form

$$\rho \approx \alpha x^3, \quad T \approx \beta x^{4/3}, \quad \alpha = [2(3 + A_1 - A_3)]^{-1/2}, \quad \beta = \alpha^{2/3} e^{2A_1/3} \tag{2.9}$$

Comparison of (2.8) and (2.9) with (1.14) and (1.15) indicates that for any A_1 as $A_2 \to \infty$ the asymptotic form changes as $x \to 0$. Existence of a limiting A_{1n} shows that passage through the adiabatic speed of sound is possible only if the ratio of the radiation pressure to gas pressure is not smaller than $A_{1n}/4 = 0.032$ at the point where the isothermal speed of sound is reached. This follows from the definition of A_1 in (1.9).

For large A_1 the solution satisfying the conditions $x = \rho = T = 0$ is obtained for

$$A_3 = A_1 + 3 - \sqrt[3]{27}\,(3A_1^{2/3})$$

In the subsonic range for large A_1 the approximate solution is of the form $T \approx x$,

$$\rho \approx x^3$$

For $0 < A_1 < \infty$ the asymptotic form for large x is

$$T \approx {}^3/_8 A_3 x (\ln x)^{-1}, \quad \rho \approx ({}^2/_3)^4 A_1 A_3^3 x^3 \ (\ln\ x)^{-4}$$

3. Solution of the problem in the general case. In order to solve system (1.11), (1.12) in the general case we must proceed from the singular point $x = \rho = T = 1$ according to expansion formulas (1.13) and integrate system (1.11), (1.12) numerically. Taking A_1 and A_2 for a single A_3, we obtain $T = \rho = 0$ when $x = 0$. Since the point $x = \rho = T = 0$ is singular, it follows that for arbitrary A_3 the solution behaves nonanalytically. For A_3 smaller than that sought, $T > 0$ for $x = 0$. For larger A_3 we have $T = 0$ for $\rho_T, x_T > 0$. In the neighborhood of the point $T = 0$, $\rho_T, x_T > 0$ system (1.11), (1.12) has the asymptotic form

$$\frac{dx}{dT} = \varphi\frac{T^3}{\rho_T}, \qquad \frac{d\rho}{dT} = \frac{\rho_T{}^3}{x_T{}^4}, \quad \varphi > 0$$

The solution is of the form

$$x = x_T + (\varphi/4\rho_T)T^4, \qquad \rho = \rho_T + T\rho_T{}^3/x_T{}^4$$

Thus, for large A_3 the integral curves $T(x)$, $\rho(x)$ lie in the domain $x > x_T$. The values of A_3 as determined by A_1 and A_2 for which the conditions $x = \rho = T = 0$ are fulfilled appear in Table 2. From Table 2 we see that for a fixed A_1 the values of A_3 change little with changes in A_2. For $A_1 > A_{1n}$ the function $A_3(A_2)$ is monotonous. Consideration of Formula (2.2) and Tables 1 and 2 yields the following observations. The derivative $(\partial A_3/\partial A_2)_{A_1}$ is positive for $A_1 > A_{1n}$ along the solutions satisfying the conditions at infinity. In the interval $3 < A_1 < 3.5$ this derivative as a function of A_2 passes through identical zero and becomes negative. In the interval $8 < A_1 < 8.5$ the derivative as a function of A_2 once again passes through identical zero and is always positive for large A_1. For $A_1 < A_{1n}$ the dependence of A_3 on A_2 is nonmonotonous. For small A_2 we find that $(\partial A_3/\partial A_2)_{A_1} > 0$; then, at $A_2 \approx 10$ for $A_1 = 0$ the derivative changes sign and remains negative all the way to $A_2 = \infty$. Where the derivative $(\partial A_3/\partial A_2)_{A_1}$ changes sign the quantity A_2 increases monotonously with A_1 and becomes infinite for $A_1 = A_{1n}$.

When $A_1 < A_{1n}$, beginning with some A_{2n} dependent on A_1 the flow at infinity is subsonic for $A_2 > A_{2n}$, while for $A_1 = A_{1n}$ we have $A_{2n} = \infty$. In the case $A_2 > A_{2n}$ the quantity A_3 is given by the relation

$$A_4 = A_2(3 + A_1 - A_3) + \frac{A_3 - 2}{A_1 + 1} = 0 \tag{3.1}$$

The total energy flux carried to infinity is equal to zero. The asymptotic form at the point $x = 0$ is of the form

$$\rho \approx \alpha x^{3/2}, \qquad T \approx \beta x, \qquad \beta = {}^2/_5\,(A_3 - 0.5/\alpha^2)$$

Here α is arbitrary. At the point A_{2n} the derivative $(\partial A_3/\partial A_2)_{A_1}$ is negative and equal

Table 2

A_1	0	0	0	0	$5\cdot10^{-4}$	$5\cdot10^{-4}$	$5\cdot10^{-4}$	$5\cdot10^{-4}$	0.01	0.01	0.01
A_2	0.1	1	10	100	0.1	1	10	100	0.1	1	10
A_3	2.84	2.945	3.056	3.012	2.84	2.945	3.056	3.012	2.845	2.955	3.06
A_1	0.01	0.1	0.1	0.1	0.1	0.1	0.1	0.5	0.5	0.5	0.5
A_2	100	0.1	1	10	20	50	100	0.1	1	10	50
A_3	3.02	2.92	3.03	3.125	3.123	3.113	3.109	3.247	3.34	3.405	3.41
A_1	1	1	1	1	1	2	3	3	5	5	5
A_2	0.1	1	5	10	50	1	1	5	0.02	0.1	0.2
A_3	3.657	3.722	3.756	3.763	3.765	4.49	5.28	5.281	6.907	6.896	6.890
A_1	5	5	7	7	10	10	10	15	15	15	15
A_2	1.0	5	0.1	1	0.02	0.1	1	0.01	0.02	0.1	0.5
A_3	6.885	6.883	8.54	8.53	11.02	11.04	11.05	15.17	15.21	15.32	15.35
A_1	20	20	50	50	50	50	100	100	100	100	100
A_2	0.02	0.2	$5\cdot10^{-4}$	0.01	0.02	0.01	10^{-4}	$5\cdot10^{-4}$	0.002	0.01	0.02
A_3	19.48	19.71	44.1	45.72	46.12	46.68	85.2	86.8	89.1	91.55	92.15
A_1	100	200	200	200	200	500	500	700	700	10^3	10^3
A_2	0.1	10^{-4}	$5\cdot10^{-4}$	0.01	0.05	10^{-4}	0.01	10^{-4}	$5\cdot10^{-3}$	10^{-4}	10^{-3}
A_3	92.95	170.7	176.2	185.2	187.1	439.5	472.2	626	664	906.5	943.5

to the corresponding quantity computed from Formula (3.1). For $A_2 > A_{2n}$ it is always the case that $A_3 > 3$. The solution with subsonic flow at infinity corresponds to the evaporation state considered in [7].

In order to apply the above solution to efflux from the corona of a red giant it is necessary to know the solution in the subsonic range. To find this solution for a known $A_3(A_1, A_2)$ it is necessary to proceed from the point $x = \rho = T = 1$ according to Formulas (1.13) and to integrate Eqs. (1.11), (1.12) for $x > 1$.

The quantity $v = 1/(\rho r^2\sqrt{T})$ is the dimensionless velocity or the ratio of the flow rate to the isothermal speed of sound. Table 3 contains the values of x, ρ, T for $v = 1/6$, 1/10,

Table 3

A_1		0.1	0.1	0.1	0.1	0.5	0.5	0.5	0.5
A_2		0.1	1	10	∞	0.1	1	10	∞
A_3		2.92	3.03	3.125	3.10	3.247	3.34	3.405	3.412
$V=\frac{1}{6}$	x	8.10	28.3	96.3	128	9.10	27.7	36.6	42.3
	ρ	154	1020	7320	$1.16\cdot10^4$	187	1090	1950	2170
	T	6.575	22.2	57.9	70.9	7.08	17.8	23.3	24.5
$V=\frac{1}{10}$	x	13.9	77.6	222	280	16.9	61.9	84.7	89.6
	ρ	581	8280	$4.56\cdot10^4$	$6.70\cdot10^4$	808	6480	$1.08.10^4$	$1.18\cdot10^4$
	T	11.1	52.7	116	137	12.6	35.1	44.2	46.1
$V=\frac{1}{20}$	x	30.2	263	596	722	42.6	166		224
	ρ	3742	$1.14\cdot10^5$	$4.34\cdot10^5$	$5.96\cdot10^5$	6740	$6.1\cdot10^4$		10^5
	T	23.9	147	268	306	29.0	80.8		101
$V=\frac{1}{30}$	x	49.9			1205		281		369
	ρ	$1.20\cdot10^4$			$2.0\cdot10^6$		$2.11\cdot10^5$		$3.29\cdot10^5$
	T	39.0			475		127		155

continued

Table 3, *continued*

A_1		1	1	1	1	5	5	5	5
A_2		0.1	1	5	∞	0.02	0.1	0.5	∞
A_3		3.657	3.722	3.756	3.767	6.907	6.896	6.888	6.883
$V=\frac{1}{6}$	x	10	22.3	24.8	27.1	7.64	9.01	9.76	10.0
	ρ	220	798	950	1099	146	191	218	228
	T	7.45	13.9	15.1	16.1	5.74	6.48	6.87	7.00
$V=\frac{1}{10}$	x	19.3	46.2	51.1	55.2	12.7	15.55	17.0	17.5
	ρ	1020	4200	4940	5610	536	747	864	907
	T	13.4	25.9	27.9	29.5	9.05	10.5	11.2	11.4
$V=\frac{1}{20}$	x	48.4	114	125	133	25.1	31.8	34.9	35.9
	ρ	8550	$3.5\cdot10^4$	$4.03\cdot10^4$	$4.48\cdot10^4$	3070	4540	5290	5555
	T	29.9	56.2	59.9	62.7	16.7	19.8	21.1	21.6
$V=\frac{1}{30}$	x	82.8		204	216	37.3	47.8	52.4	53.9
	ρ	$2.98\cdot10^4$		$1.3\cdot10^5$	$1.43\cdot10^5$	8530	$1.28\cdot10^4$	$1.49\cdot10^4$	$1.57\cdot10^4$
	T	47.7		91.4	95.3	23.95	28.5	30.4	31.0

1/20, 1/30 for certain values of A_1, A_2.

For $A_2 = 0$ we have $A_3 = 0.8186(1 + A_1) + 2$; given below are the values of the dimensionless solution $\rho(x)$ and $T(x)$ (the same for all A_1) for various values of $1/v$ and x:

Table 4

$1/v = 2$	4	6	8	10	12	14
$x = 2.32$	4.59	6.53	8.27	9.88	11.4	12.8
$\rho = 7.59$	43.9	114	218	358	533	743
$T = 2.02$	3.68	5.07	6.31	7.45	8.51	9.51
$1/v = 16$	18	20	26	30	36	40
$x = 14.2$	15.4	16.7	20.2	22.4	25.5	27.5
$\rho = 991$	1270	1590	2770	3735	5460	6800
$T = 10.5$	11.4	12.3	14.8	16.3	18.5	19.9
$1/v = 46$	50	60	70	80	90	100
$x = 30.4$	32.2	36.6	40.8	44.7	48.5	52.2
$\rho = 9080$	$1.08\cdot10^4$	$1.57\cdot10^4$	$2.15\cdot10^4$	$2.83\cdot10^4$	$3.60\cdot10^4$	$4.46\cdot10^4$
$T = 21.9$	23.2	26.3	29.2	32.0	34.7	37.3

If the initial conditions (e.g. in the stellar corona) are known, then the resulting solution enables one to determine the mass flux. A complete solution requires knowledge of three of the following quantities: L, r_0, T_0, ρ_0, u_0. The remaining two quantities are not independent. Expressing these dimensional quantities in terms of dimensionless quantities and parameters, we obtain

$$L = \frac{4\pi c GM}{\varkappa}\frac{A_1 A_4}{A_3}$$

$$r_0 = \left(\frac{4\sigma\varkappa}{3c}\right)^{2/5}\frac{(GM)^{7/5}}{R^{9/5}}\frac{1}{(A_1^2A_2A_3)^{2/5}A_3x}, \qquad \rho_0 = \left(\frac{3R}{4\sigma}\right)^{1/5}\left(\frac{c\sqrt{R}}{\varkappa GM}\right)^{6/5}(A_1^2A_2A_3)^{6/5}\frac{\rho}{A_1} \tag{3.2}$$

$$T_0 = \left(\frac{3cR^{3/2}}{4\sigma\varkappa GM}\right)^{2/5}(A_1^2A_2A_3)^{2/5}T, \qquad u_0 = R^{4/5}\left(\frac{3c}{4GM\sigma\varkappa}\right)^{1/5}(A_1^2A_2A_3)^{1/5}\frac{x^2}{\rho}$$

Determining the dimensionless parameters A_1, A_2, x from three quantities (e.g., L, r_0, u_0), making use of (1.10) and (2.2), and Tables 1 to 4, we obtain the initial values ρ_0 and T_0 and determined the mass flux,

$$4\pi\mu = 4\pi\frac{(GM)^{7/5}}{R^{9/5}}\left(\frac{4\sigma}{3}\right)^{2/5}\left(\frac{c}{\varkappa}\right)^{3/5}\frac{(A_1^2A_2A_3)^{3/5}}{A_1A_3^2} \tag{3.3}$$

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THE LIMITING MOISTURE PROFILE DURING INFILTRATION IN TO A HOMOGENEOUS SOIL

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The present paper deals with a quasilinear second order parabolic equation describing an unsteady one-dimensional infiltration and investigates the time asymptotic of the solution of the proble m of formation of moisture saturation profile when the infiltration starts at the surface. The existence of a limiting profile expanding with a constant velocity is proved and estimates are given for the speed of approach to this profile with increasing time, when the soil has unlimited capacity. An estimate of the speed of approach to the steady (homogeneous) distribution is also given for the soil of limited capacity.

During the infiltration into a homogeneous soil, moisture $u(t, x)$ of the soil being a function of time t and of depth x of the layer (the X-axis is directed downwards), satisfies an equation of the type

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] - \frac{\partial K(u)}{\partial x} \tag{1}$$

$$D(u) > 0, \quad K(u) > 0, \quad D'(u) > 0, \quad K'(u) > 0, \quad K''(u) \geqslant \mu > 0 \text{ when } (u \geqslant u_0 > 0)$$

Taking into account initial moisture distribution in the soil and infiltration on the surface of the ground, we obtain the following boundary condition:

$$u(t, 0) = u_1 \quad (t > 0), \qquad u(0, x) = u_0(x) \quad (0 \leqslant x < \infty)$$

$$u_0 \leqslant u_0(x) \leqslant u_1 \quad (\lim u_0(x) = u_0) \text{ when } x \to +\infty) \tag{2}$$

Here $u_1 = 1$ denotes the moisture corresponding to full saturation of soil on the earth surface.

In the presence of ground water at the depth $x = X$, our boundary condition assumes the form

$$u(t, 0) = u_1, \ u(t, X) = u_1, \ u(0, x) = u_0(x)$$

$$0 \leqslant x \leqslant X, \ u_0 \leqslant u_0(x) \leqslant u_1 \tag{3}$$

The problem of determination of the limiting moisture profile during infiltration into the